The eigenvalue spectrum of a large symmetric random matrix

1976 J. Phys. A: Math. Gen. 9 1595

(http://iopscience.iop.org/0305-4470/9/10/011)

J. Phys. A: Math. Gen., Vol. 9, No. 10, 1976. Printed in Great Britain. © 1976

The eigenvalue spectrum of a large symmetric random matrix

S F Edwards†‡ and Raymond C Jones§

† Science Research Council, State House, High Holborn, London WC1R 4TA, UK
§ Department of Mathematical Physics, University of Birmingham, Birmingham B15 2TT, UK

Received 21 May 1976

Abstract. A new and straightforward method is presented for calculating the eigenvalue spectrum of a large symmetric square matrix each of whose upper triangular elements is described by a Gaussian probability density function with the same mean and variance. Using the $n \to 0$ method, we derive the semicircular eigenvalue spectrum when the mean of each element is zero and show that there is a critical finite mean value above which a single eigenvalue splits off from the semicircular continuum of eigenvalues.

1. Introduction

A recent model of the thermodynamics of a 'spin glass' proposed by Sherrington and Kirkpatrick (1975) is similar to the Kac model of ferromagnetism (see e.g. Stanley 1971) in which each spin is allowed to interact with every other with an interaction strength which is a Gaussian random variable with non-zero mean. Two papers since that time have emphasized the equivalence between this approach and a study of the averaged eigenvalue spectrum of a large symmetric random matrix in which each element is a Gaussian random variable (Thouless *et al* 1976, Kosterlitz *et al* 1976).

The averaged eigenvalue spectrum of a large $N \times N$ Hermitian matrix each of whose elements has a Gaussian probability density function with mean zero and fixed variance seems first to have been given by Wigner (unpublished—see, however, Bronk 1964) and is known as the semicircular law. A published derivation of this law by Mehta (1967) is difficult to follow and the answer quoted is confusing in both the width of the distribution and its normalization.

In this short paper we present a simple derivation of this semicircular law valid as $N \to \infty$, and use the $n \to 0$ trick exploited by Edwards (1970, 1971) and Edwards and Anderson (1975), in which one writes

$$\ln x \equiv \lim_{n \to 0} \left(\frac{x^n - 1}{n} \right). \tag{1.1}$$

In § 2 we set up the formalism necessary for calculating the eigenvalue spectrum and illustrate the method by calculating the spectrum of a matrix each of whose elements is the same and constant.

In § 3 we use this formalism to calculate the (semicircular) averaged eigenvalue spectrum of a real symmetric $N \times N$ matrix each of whose elements has zero mean and

‡ Also at Cavendish Laboratory, Cambridge University, Madingley Road, Cambridge, UK.

finite variance. Finally in § 4 we calculate the coupled spectrum of a random matrix each of whose elements has finite mean and we point out certain similarities between this and certain problems in solid state physics involving localized perturbations. The analysis is trivially extensible to Hermitian matrices.

2. The technique

Consider a real symmetric $N\times N$ matrix $\mathbf{M}$ with eigenvalues M_λ. This will have an eigenvalue spectrum

$$\nu(\lambda)=N^{-1}\sum_\lambda \delta(\lambda-M_\lambda) \tag{2.1}$$

where $\nu(\lambda)$ is normalized to unity.

If λ is given a small negative imaginary part $-\mathrm{i}\epsilon$, then we see that

$$\nu(\lambda)=(\pi N)^{-1}\,\mathrm{Im}(\lambda-\mathrm{i}\epsilon-M_\lambda)^{-1}. \tag{2.2}$$

Now

$$\det(\mathbf{1}\lambda-\mathbf{M})\equiv\prod_\lambda(\lambda-M_\lambda).$$

Hence

$$\nu(\lambda)\equiv\frac{1}{N\pi}\,\mathrm{Im}\,\frac{\partial}{\partial\lambda}\ln\det(\mathbf{1}(\lambda-\mathrm{i}\epsilon)-\mathbf{M}) \tag{2.3}$$

and using the representation of the logarithm given by (1.1) we obtain

$$\nu(\lambda)\equiv\frac{-2}{N\pi}\,\mathrm{Im}\,\frac{\partial}{\partial\lambda}\lim_{n\to0}\frac{1}{n}[(\det{}^{-1/2}(\mathbf{1}\lambda-\mathbf{M}))^n-1]. \tag{2.4}$$

(We shall not explicitly display the small imaginary part of λ, but leave this understood.)

The determinant of a symmetric matrix may be represented by the multiple Fresnel integral

$$\det{}^{-1/2}(\mathbf{1}\lambda-\mathbf{M})=\left(\frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\pi^{1/2}}\right)^N\int_{-\infty}^{\infty}\prod\mathrm{d}x_i\,\exp\Big(-\mathrm{i}\sum_{ij}x_i(\mathbf{1}\lambda-\mathbf{M})_{ij}x_j\Big) \tag{2.5}$$

and the imaginary part of λ ensures convergence. We now substitute (2.5) into (2.4), assume that the latter holds for integral values of n and may be continued to $n=0$. Thus we obtain our basic result

$$\nu(\lambda)=\frac{2}{N\pi}\,\mathrm{Im}\,\frac{\partial}{\partial\lambda}\lim_{n\to0}\frac{1}{n}\left[\left(\frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\pi^{1/2}}\right)^{Nn}\int_{-\infty}^{\infty}\prod\mathrm{d}x_i\,\exp\Big(-\mathrm{i}\sum_{ij;\alpha}x_i^\alpha(\lambda\delta_{ij}-M_{ij})x_j^\alpha\Big)-1\right]. \tag{2.6}$$

The integration is now over the Nn variables x_i^α and the limit $n\to0$ is taken first.

We give first a simple illustrative example of the use of (2.6) when each matrix element M_{ij} has the constant value M_0/N and we choose M_0 to be of order unity. Using the auxiliary field identity

$$\exp\left[\mathrm{i}\frac{M_0}{N}\Big(\sum_i x_i^\alpha\Big)^2\right]\equiv\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\mathrm{d}q\,\exp\Big(\frac{\mathrm{i}q^2}{2}\Big)\exp\left[-\Big(\frac{2M_0}{N}\Big)^{1/2}q\sum_i x_i^\alpha\right] \tag{2.7}$$

we can rewrite the integral in (2.6) as

$$J_1 \equiv \prod_\alpha \frac{e^{-i\pi/4}}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dq \prod dx_i^\alpha \exp\left[-i\lambda \sum_i (x_i^\alpha)^2 - \left(\frac{2M}{N}\right)^{1/2} q \sum_i x_i^\alpha + \frac{iq^2}{2}\right]. \tag{2.8}$$

The integrals over the x_i^α are easily performed by completing the square and we find

$$J_1 \equiv \prod_\alpha \frac{e^{-i\pi/4}}{(2\pi)^{1/2}} \exp\left(-\frac{iN\pi}{4}\right)\left(\frac{\pi}{\lambda}\right)^{N/2} \int_{-\infty}^{\infty} dq \exp\left(\frac{iq^2}{2}\right) \exp\left(-\frac{iM_0 q^2}{2\lambda}\right) \tag{2.9}$$

which gives

$$J_1 \equiv \prod_\alpha \exp\left(\frac{-iN\pi}{4}\right) \pi^{N/2} \lambda^{-(N-1)/2} (\lambda - M_0)^{-1/2}. \tag{2.10}$$

Substituting this back into (2.6) and using the identity (1.1) we obtain

$$\nu(\lambda) = \frac{-2}{N\pi} \operatorname{Im} \frac{\partial}{\partial\lambda} \ln\left(\lambda^{-(N-1)/2}(\lambda - M_0)^{-1/2}\right). \tag{2.11}$$

Differentiating with respect to λ and using $\operatorname{Im} \lambda^{-1} = \pi\delta(\lambda)$, we obtain

$$\nu(\lambda) = \left(\frac{N-1}{N}\right)\delta(\lambda) + \frac{1}{N}\delta(\lambda - M_0) \tag{2.12}$$

which is of course the correct spectrum of the degenerate matrix with $N-1$ eigenvalues zero and one with value M_0.

3. The semicircular law

We consider a real symmetric $N \times N$ matrix $\mathbf{M}$ (with $M_{ij} = M_{ji}$) and allow each matrix element to fluctuate about a zero mean. The probability density function of the element M_{ij} we choose as the Gaussian

$$p(M_{ij}) = [\exp(-M_{ij}^2/2\sigma^2)]/(2\pi\sigma^2)^{1/2}. \tag{3.1}$$

For convenience we define J by $\sigma^2 \equiv J^2/N$ and J will be of order unity.

The averaged density of eigenvalues $\rho(\lambda)$ is obtained by ensemble averaging equation (2.6) for $\nu(\lambda; \{M_{ij}\})$ over all configurations of the M_{ij} given by (3.1). Thus

$$\rho(\lambda) = \int \nu(\lambda; \{M_{ij}\}) \prod p(M_{ij})\, dM_{ij}. \tag{3.2}$$

Carrying out the Gaussian integrations we find in a straightforward manner that

$$\rho(\lambda) = \frac{-2}{N\pi} \operatorname{Im} \frac{\partial}{\partial\lambda} \lim_{n\to 0} \frac{1}{n}\left[\!\left[\left(\frac{e^{i\pi/4}}{\pi^{1/2}}\right)^{Nn} \int_{-\infty}^{\infty} \prod dx_i^\alpha \left\{\exp\left(-i\lambda \sum_{i,\alpha} (x_i^\alpha)^2\right)\right.\right.\right.$$

$$\left.\left.\left.\times \exp\left[\frac{-J^2}{N} \sum_{i,j} \left(\sum_\alpha x_i^\alpha x_j^\alpha\right)^2\right] \exp\left[\frac{J^2}{N} \sum_i \left(\sum_\alpha (x_i^\alpha)^2\right)^2\right]\right\} - 1\right]\!\right]. \tag{3.3}$$

Since we shall ultimately want the leading-order terms in N, and the terms linear in n (as $n \to 0$) in the exponent of (3.3) we must examine carefully the order of magnitude of the second two terms in the exponential of this equation. The second term is

$$\frac{J^2}{N}\sum_{ij}\Big(\sum_{\alpha} x_i^{\alpha}x_j^{\alpha}\Big)^2 \equiv \frac{J^2}{N}\sum_{\substack{\alpha\beta\\ ij}} x_i^{\alpha}x_j^{\alpha}x_i^{\beta}x_j^{\beta}$$

and this may be written as

$$\frac{J^2}{N}\sum_{\alpha}\Big(\sum_{i}(x_i^{\alpha})^2\Big)^2 + \frac{J^2}{N}\sum_{\substack{\alpha\neq\beta\\ ij}} x_i^{\alpha}x_j^{\alpha}x_i^{\beta}x_j^{\beta}. \tag{3.4}$$

The first term of (3.4) is of order Nn, whilst the second has a zero mean but its square is of order n. The third term in the exponential of (3.3) is only of order n^2 and hence we need only retain the terms in $\alpha=\beta$ from (3.4). Thus

$$\rho(\lambda)=\frac{-2}{N\pi}\operatorname{Im}\frac{\partial}{\partial\lambda}\lim_{n\to 0}\frac{1}{n}\left[\!\left[\left\{\left(\frac{e^{i\pi/4}}{\pi^{1/2}}\right)^{Nn}\int_{-\infty}^{\infty}\prod \mathrm{d}x_i \exp\left[-i\lambda\sum_i x_i^2-\frac{J^2}{N}\Big(\sum_i x_i^2\Big)^2\right]\right\}^n-1\right]\!\right] \tag{3.5}$$

for large N. Again we use an auxiliary field s, to parametrize the second exponential as

$$\exp\left[\frac{-J^2}{N}\Big(\sum_i (x_i)^2\Big)^2\right]\equiv\left(\frac{N}{2\pi}\right)^{1/2}\frac{\lambda}{(2J^2)^{1/2}}\int_{-\infty}^{\infty}\mathrm{d}s\,\exp\left(\frac{-\lambda^2}{4J^2}Ns^2\right)\exp\Big(-i\lambda s\sum_i (x_i)^2\Big). \tag{3.6}$$

The integral in (3.5), which we denote by J_2, then becomes

$$J_2=\left[\int_{-\infty}^{\infty}\mathrm{d}s\prod \mathrm{d}x_i\left(\frac{N}{2\pi}\right)^{1/2}\frac{\lambda}{(2J^2)^{1/2}}\exp\Big(-i\lambda(1+s)\sum_i (x_i)^2\Big)\exp\left(\frac{-N\lambda^2s^2}{4J^2}\right)\right]^n. \tag{3.7}$$

The integrals over the $\{x_i\}$ are straightforward Fresnel integrals† and yield

$$J_2=\left[\left(\frac{N}{2\pi}\right)^{1/2}\left(\frac{\pi^N}{2J^2}\right)^{1/2}\lambda\,\exp\left(\frac{-N}{2}\ln\lambda\right)\int_{-\infty}^{\infty}\mathrm{d}s\,\exp(-Ng(s))\right]^n \tag{3.8}$$

where

$$g(s)=\frac{\lambda^2s^2}{4J^2}+\tfrac{1}{2}\ln[i(1+s)]. \tag{3.9}$$

It should be noted that the small imaginary part of λ defines the position of the branch point to be in the upper half-plane at $s=-1$. We may then cut the complex s plane by a line running from -1 to $-\infty$ lying above the real axis and the contour of integration in (3.7) will be well defined. Since we require an expression for the eigenvalue spectrum valid as $N\to\infty$ we shall ultimately only retain terms in (3.8) which are of order e^N and the integral can then be evaluated by saddle-point integration. The saddle points of (3.9) occur when $g'(s)=0$ and for $|\lambda|<2J$ are easily seen to lie at the conjugate points

$$s_0^{\pm}=\frac{1}{2}\left[-1\pm i\left(\frac{4J^2}{\lambda^2}-1\right)^{1/2}\right] \tag{3.10}$$

whilst for $|\lambda|>2J$ we have real saddle points at

$$s^{\pm}=\frac{1}{2}\left[-1\pm\left(1-\frac{4J^2}{\lambda^2}\right)^{1/2}\right]. \tag{3.11}$$

† An alternative approach is to use polar coordinates in the space of the x_i when by steepest descent ρ is a multiple of Im R^2.

We consider first the case $|\lambda|<2J$. In order that the conditions for saddle-point integration apply, a contour must be chosen along which $\operatorname{Re} g(s)$ is minimum at the saddle point. It may be verified that if the contour of integration is deformed downwards to follow the line $s = x - \mathrm{i}[(4J^2/\lambda^2)-1]^{1/2}$, $\operatorname{Re} g(s)$ has a global minimum at $x = -\frac{1}{2}$ corresponding to the saddle point s_0^-. However along the line $\operatorname{Re} s = -\frac{1}{2}$, we find that $\operatorname{Re} g(s)$ has maxima at the points $s_0^{\pm}$ and a minimum at $s = -\frac{1}{2}$. Thus if we choose the contour to pass through s_0^-, it cannot be deformed to pass through s_0^+ also since it would cross regions in which the integrand is larger than at the saddle point and the conditions for using saddle-point integration to obtain the asymptotic behaviour of (3.8) would not be obtained. For $|\lambda|<2J$ we use the contour shown in figure 1.

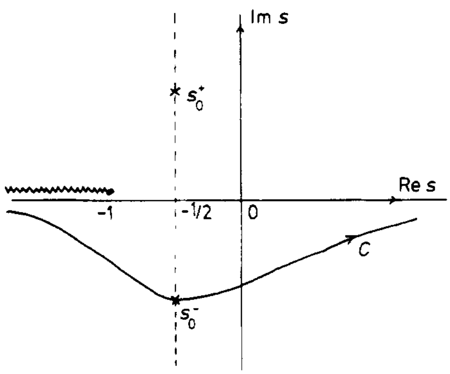

Figure 1.

We thus find that

$$\int_{-\infty}^{\infty} \mathrm{d}s \exp(-Ng(s)) \approx \frac{\exp(-Ng(s_0^-))(2\pi)^{1/2}\exp(\mathrm{i}\psi^-)}{|Ng''(s_0^-)|^{1/2}} \tag{3.12}$$

where ψ^- is the direction of crossing the saddle point and is given by $\psi^- = -\frac{1}{2}\tan^{-1}\{-|[(4J^2/\lambda^2)-1]^{1/2}|^{-1}\}$. Since $g''(s_0^-)$ is of order unity we need only retain the exponential in (3.12). Substituting (3.12), (3.10), and (3.8) into (3.5) and making use of (1.1) immediately gives

$$\rho(\lambda) = \frac{-2}{N\pi}\operatorname{Im}\frac{\partial}{\partial\lambda}\left(-Ng(s_0^-) - \frac{N}{2}\ln\lambda\right). \tag{3.13}$$

We now use (3.10) for s_0^-, then a simple differentiation yields

$$\rho(\lambda) \equiv \rho_0(\lambda) \equiv \frac{1}{2\pi J^2}(4J^2-\lambda^2)^{1/2}, \qquad |\lambda|<2J. \tag{3.14}$$

This is the semicircular law for the density of eigenvalues of a large random symmetric matrix. The result differs from that given by Mehta (1967, p 240) the edges of whose bands appear at $\lambda = \pm J\sqrt{2}$. However, it is easily verified that (3.14) is correctly normalized to unity and correctly gives a second moment of a random symmetric matrix in agreement with a first-principles calculation.

For $|\lambda|>2J$, $g(s_0^-)$ is real and hence $\rho(\lambda)=0$.

It has been pointed out (Bronk 1964, Mehta 1967) that if the matrix is large but finite, the spectrum will have an exponential tail of states with a finite number of eigenvalues concentrated in a region $O(N^{-1/6})$ beyond $2J$.

4. The spectrum when each element has a finite mean

We now consider a large symmetric random matrix each of whose elements fluctuates about a fixed mean M_0/N with a probability density function

$$p(M_{ij})=(2\pi\sigma^2)^{-1/2}\exp\left[-\left(M_{ij}-\frac{M_0}{N}\right)^2\frac{1}{2\sigma^2}\right]. \tag{4.1}$$

Again we define $J^2=N\sigma^2$, where J is a number of order unity. We proceed as before by using (3.2) and (4.1) to calculate the averaged eigenvalue spectrum. The Gaussian integrals over the matrix elements are easily calculated, and after retaining the dominant terms in N and n in the exponent we find that

$$\rho(\lambda)=\frac{-2}{N\pi}\,\mathrm{Im}\,\frac{\partial}{\partial\lambda}\lim_{n\to 0}\frac{1}{n}\left[\!\!\left[\left(\frac{e^{i\pi/4}}{\pi^{1/2}}\right)^{Nn}\int_{-\infty}^{\infty}\prod dx_i^\alpha\left\{\exp\left(-i\lambda\sum_{i,\alpha}(x_i^\alpha)^2\right)\right.\right.\right.$$
$$\left.\left.\left.\times\exp\left[\frac{-J^2}{N}\sum_\alpha\left(\sum_i(x_i^\alpha)^2\right)^2\right]\exp\left[\frac{iM_0}{N}\sum_\alpha\left(\sum_i x_i^\alpha\right)^2\right]\right\}-1\right]\!\!\right]. \tag{4.2}$$

We denote the integral in (4.2) by J_3, and it may be parametrized using the auxiliary field identities (2.7) and (3.7). J_3 then becomes

$$J_3=\prod_\alpha\frac{e^{-i\pi/4}}{2\pi}\frac{\lambda N}{(4M_0J^2)^{1/2}}\int_{-\infty}^{\infty}ds\,dq\prod dx_i^\alpha\exp\left(-i\lambda(1+s)\sum_i(x_i^\alpha)^2\right)$$
$$\times\exp\left(-q\sum_i x_i^\alpha\right)\exp\left(\frac{-\lambda^2N}{4J^2}s^2\right)\exp\left(\frac{iNq^2}{4M_0}\right). \tag{4.3}$$

The Gaussian integration over the $\{x_i^\alpha\}$ is performed straightforwardly by completing the square and we obtain

$$J_3=\prod_\alpha\frac{e^{-i\pi/4}}{2\pi}\frac{\lambda N\pi^{N/2}}{(4M_0J^2)^{1/2}}\exp\left(\frac{-N}{2}\ln\lambda\right)\int ds\,dq\exp(-Ng(s))$$
$$\times\exp\left[\frac{-iNq^2}{4}\left(\frac{1}{\lambda(1+s)}-\frac{1}{M_0}\right)\right]. \tag{4.4}$$

We now perform the remaining Gaussian integral over q and obtain the result

$$J_3=\prod_\alpha\frac{e^{-i\pi/4}}{2\pi}\frac{\lambda N\pi^{N/2}}{(4M_0J^2)^{1/2}}\exp\left(\frac{-N}{2}\ln\lambda\right)\left(\frac{4\pi M_0}{N}\right)^{1/2}\int ds(1+s)^{1/2}\frac{\exp(-Ng(s))}{[i(s_1-s)]^{1/2}} \tag{4.5}$$

where

$$s_1\equiv-1+(M_0/\lambda). \tag{4.6}$$

Now away from regions in which $s \approx s_1$ the square root singularity in the denominator of (4.5) can be neglected because of the N in the exponential. The leading-order asymptotic contribution to the integral is then again given by the saddle point s_0^- of $g(s)$ and we obtain the semicircular law $\rho_0(\lambda)$ for the eigenvalue density once more (alternatively one easily sees that if s is not close to s_1, and the functions pre-multiplying the exponential are re-exponentiated themselves, the resulting function has a saddle point only shifted by $O(1/N)$ from s_0^-).

Close to s_1 more care is required and the arguments we use are close to those used in the spherical model of magnetism (Berlin and Kac 1958). A Taylor expansion of $g(s)$ about s_1 gives

$$g(s) \approx g(s_1) + \frac{\lambda}{2J^2}(\lambda_m - \lambda)(s - s_1) + \frac{\lambda^2}{4}\left(\frac{1}{J^2} - \frac{1}{M_0^2}\right)\frac{(s - s_1)^2}{2!} + \dots \tag{4.7}$$

where the important quantity, λ_m, is defined by

$$\lambda_m = M_0 + \frac{J^2}{M_0}. \tag{4.8}$$

Thus when $\lambda = \lambda_m$, $g(s)$ has a turning point at $s = s_1$: for $M_0 > J$ the turning point is a minimum whilst for $M_0 < J$ it is a maximum. The former then leads to a local maximum in the integral whose contribution to the integrand for this isolated value $\lambda = \lambda_m$ must be included. The case $M_0 < J$ leads to a minimum of the integrand which is of no importance. It should be noticed that the two cases $M_0 \gtrless J$ correspond to $\lambda_m \gtrless 2J$, i.e. we only include the contribution of s_1 when λ_m lies outside the semicircular continuum of eigenvalues. In this latter case the integral (4.5) is evaluated straightforwardly by retaining the first two terms of the Taylor expansion of $g(s)$ and using the substitution $s - s_1 = iu^2$ to obtain a straightforward Gaussian integral. The result is

$$J_3 = \prod_\alpha \left(\frac{2M_0}{J^2}\right)^{1/2} \pi^{N/2} \exp\left(\frac{-N}{2}\ln\lambda\right) \exp(-Ng(s_1)) \exp[-\tfrac{1}{2}\ln(\lambda_m - \lambda)]. \tag{4.9}$$

Now

$$g(s_1) = \frac{-\lambda^2}{4J^2}\left(\frac{M_0}{\lambda} - 1\right)^2 + \frac{\ln\lambda}{2} - \frac{\ln M_0}{2} - \frac{i\pi}{4} \tag{4.10}$$

thus

$$J_3 = \prod_\alpha \left(\frac{2M_0}{J^2}\right)^{1/2} \pi^{N/2} \exp\left(\frac{-i\pi}{4}\right) \exp\left(\frac{-N}{2}\ln M_0\right) \exp\left[\frac{-\lambda^2 N}{4J^2}\left(\frac{M_0}{\lambda} - 1\right)^2 - \tfrac{1}{2}\ln(\lambda_m - \lambda)\right]. \tag{4.11}$$

This is now substituted back into (4.2) whence, using (1.1), we find that for $M_0 > J$ the contribution to the eigenvalue spectrum is

$$\rho(\lambda) = \frac{-2}{N\pi}\,\mathrm{Im}\,\frac{\partial}{\partial\lambda}\left[-\tfrac{1}{2}\ln(\lambda_m - \lambda)\right] = \frac{1}{N}\delta(\lambda - \lambda_m) \tag{4.12}$$

with no corresponding contribution for $M_0 < J$.

Thus when each element of a large symmetric random matrix has the same mean, the eigenvalue spectrum is

$$\rho(\lambda)=\begin{cases}\rho_0(\lambda)+\dfrac{1}{N}\delta\left[\lambda-\left(M_0+\dfrac{J^2}{M_0}\right)\right] & |M_0|>J\\ \rho_0(\lambda) & |M_0|<J.\end{cases} \tag{4.13}$$

This is the result we wished to establish.

As was pointed out in a recent paper (Kosterlitz *et al* 1976) this calculation is analogous to the extensively studied problem in solid state physics of determining the eigenvalue spectrum of a system which contains a strongly coupled localized perturbation (see e.g. Izyumov 1965 for a review). It is well known that for certain values of the coupling constants of the system, a state may be split outside the continuum of wave-like states and contribute a delta function, outside the band, to the eigenvalue spectrum. The wavefunction of such a state is highly localized. This case is analogous to our limit $M_0>J$. For different values of the coupling constants this state will have an energy lying inside that of the lost band with which a weak resonance is associated. This corresponds to $M_0<J$.

5. Summary

We have considered a large random symmetric matrix the upper triangular elements of which are independent Gaussian random variables with mean M_0/N and variance J^2/N; our results are true in the limit $N\to 0$. We have used the $n\to 0$ method to evaluate the eigenvalue density and find

(i)
$$\rho_0(\lambda)=\begin{cases}\dfrac{(4J^2-\lambda^2)^{1/2}}{2\pi J^2}; & M_0=0;\ |\lambda|<2J\\ 0; & M_0=0;\ |\lambda|>2J\end{cases}$$

(ii)
$$\rho(\lambda)=\begin{cases}\rho_0(\lambda)+N^{-1}\delta\{\lambda-[M_0+(J^2/M_0)]\}; & |M_0|>J\\ \rho_0(\lambda); & |M_0|<J\end{cases}$$

Acknowledgments

The authors are pleased to acknowledge extended and helpful discussions with Professors T H R Skyrme and D J Thouless and with Dr J M Kosterlitz.

References

Berlin T H and Kac M 1952 *Phys. Rev.* **86** 821–35
Bronk B V 1964 *J. Math. Phys.* **5** 215–20
Edwards S F 1970 *Proc. 4th Int. Conf. on Amorphous Materials* eds R W Douglas and B Ellis (New York: Wiley)
—— 1971 *Polymer Networks* eds A J Chompff and S Newman (New York: Plenum Press)
Edwards S F and Anderson P W 1975 *J. Phys. F: Metal Phys.* **5** 965–74

Izyumov Y A 1965 *Adv. Phys.* **14** 569–619

Kosterlitz J M, Thouless D J and Jones R C 1976 *Phys. Rev. Lett.* **36** 1217–20

Mehta M L 1967 *Random Matrices and the Statistical Theory of Energy Levels* (London and New York: Academic Press) p 240

Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792–6

Stanley H E 1971 *Phase Transitions and Critical Phenomena* (New York: Oxford University Press) p 91

Thouless D J, Anderson P W and Palmer R G 1976 *Princeton University Preprint*